

A new weak type estimate for Marcinkiewicz integrals on weighted Hardy spaces

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Abstract

The Marcinkiewicz integral μ_Ω is essentially a Littlewood-Paley g -function, which plays a important role in harmonic analysis. In this paper, by using the atomic decomposition theory of weighted Hardy spaces, we will obtain the weighted weak type estimate of μ_Ω on these spaces, under some Dini type condition imposed on the kernel Ω . This result is new even in the unweighted case.

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1 Introduction

Suppose that S^{n-1} is the unit sphere in \mathbb{R}^n ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma$. Let Ω be a homogeneous function of degree zero on \mathbb{R}^n satisfying $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. Then the Marcinkiewicz integral operator μ_Ω of higher dimension is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.$$

This operator μ_Ω was first defined by Stein in [15]. He proved that if $\Omega \in Lip_\alpha(S^{n-1})$ ($0 < \alpha \leq 1$), then μ_Ω is the operator of strong type (p, p) for $1 < p \leq 2$ and of weak type $(1, 1)$. It is well known that the Littlewood-Paley g -function

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is a very important tool in harmonic analysis and the Marcinkiewicz integral is essentially a Littlewood-Paley g -function. Therefore, many authors have been interested in studying the boundedness properties of μ_Ω on various function spaces. In [1], Benedek, Calderón and Panzone showed that if Ω is continuously differentiable on S^{n-1} then μ_Ω is of strong type (p, p) for $1 < p < \infty$. In 1990, Torchinsky and Wang [18] considered the weighted case and proved the following theorem.

Theorem A. *Let $\Omega \in Lip_\alpha(S^{n-1})$, $0 < \alpha \leq 1$. If $w \in A_p$ (Muckenhoupt weight class), $1 < p < \infty$, then there exists a constant $C > 0$ independent of f such that*

$$\|\mu_\Omega(f)\|_{L_w^p} \leq C\|f\|_{L_w^p}.$$

Here, and in what follows, we shall denote the conjugate exponent of $q > 1$ by $q' = q/(q-1)$. In 1999, Ding, Fan and Pan [3] improved the result mentioned above by ridding of the smoothness condition imposed on Ω .

Theorem B. *Let $\Omega \in L^q(S^{n-1})$, $1 < q < \infty$. If $w^{q'} \in A_p$, $1 < p < \infty$, then there is a constant $C > 0$ independent of f such that*

$$\|\mu_\Omega(f)\|_{L_w^p} \leq C\|f\|_{L_w^p}.$$

In order to obtain the $H^p(H_w^p)$ boundedness of μ_Ω , we need to introduce the notion of $L^{q,\alpha}$ -Dini condition. For $q \geq 1$ and $0 \leq \alpha \leq 1$, we say that Ω satisfies the $L^{q,\alpha}$ -Dini condition if $\Omega \in L^q(S^{n-1})$ is homogeneous of degree zero on \mathbb{R}^n and

$$\int_0^1 \frac{\omega_q(\delta)}{\delta^{1+\alpha}} d\delta < \infty, \quad 0 \leq \alpha \leq 1,$$

where $\omega_q(\delta)$ denotes the integral modulus of continuity of order q for Ω defined by

$$\omega_q(\delta) = \sup_{|\rho| < \delta} \left(\int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^q d\sigma(x') \right)^{1/q}$$

and ρ is a rotation in \mathbb{R}^n with $|\rho| = \|\rho - I\|$. When $\alpha = 0$, it is called the L^q -Dini condition. For $0 < \beta < \alpha \leq 1$, if Ω satisfies the $L^{q,\alpha}$ -Dini condition, then it also satisfies the $L^{q,\beta}$ -Dini condition. We thus denote by $\text{Din}_\alpha^q(S^{n-1})$ the class of all functions which satisfy the $L^{q,\beta}$ -Dini condition for all $0 < \beta < \alpha$.

In 2002, Ding, Lu and Xue [5] showed that if Ω satisfies the L^1 -Dini condition, then μ_Ω is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. One year later, Ding, Lee and Lin [4] extended it to the weighted case.

Theorem C. *Let Ω satisfy the L^q -Dini condition for $q > 1$. If $w^{q'} \in A_1$, then there exists a constant $C > 0$ independent of f such that*

$$\|\mu_\Omega(f)\|_{L_w^1} \leq C\|f\|_{H_w^1}.$$

Theorem D. *Let $0 < \alpha \leq 1$, $\beta = \min\{\alpha, 1/2\}$ and $n/(n+\beta) < p < 1$. If $\Omega \in Lip_\alpha(S^{n-1})$ and $w \in A_{p+\frac{p\beta}{n}}$, then there exists a constant $C > 0$ independent of f such that*

$$\|\mu_\Omega(f)\|_{L_w^p} \leq C\|f\|_{H_w^p}.$$

In 2007, Lin and Lin [12] proved that under weaker smoothness conditions assumed on Ω ; that is $\Omega \in \text{Din}_\alpha^q(S^{n-1})$, then μ_Ω is also bounded from $H_w^p(\mathbb{R}^n)$ into $L_w^p(\mathbb{R}^n)$. More precisely, they proved

Theorem E. *Let $0 < \alpha \leq 1$, $\beta = \min\{\alpha, 1/2\}$ and $n/(n + \beta) < p \leq 1$. Suppose that $\Omega \in \text{Din}_\alpha^q(S^{n-1})$ for $q > 1$. If $w^{q'} \in A_{(p + \frac{p\beta}{n} - \frac{1}{q})q'}$, then there exists a constant $C > 0$ independent of f such that*

$$\|\mu_\Omega(f)\|_{L_w^p} \leq C\|f\|_{H_w^p}.$$

The main goal of this article is to study the weak type estimates of μ_Ω on the weighted Hardy spaces $H_w^p(\mathbb{R}^n)$ at the endpoint case of $p = n/(n + \beta)$ and $\beta = \min\{\alpha, 1/2\}$. We now present our main result as follows.

Theorem 1.1. *Let $0 < \alpha \leq 1$, $\beta = \min\{\alpha, 1/2\}$ and $p = n/(n + \beta)$. Suppose that $\Omega \in \text{Din}_\alpha^q(S^{n-1})$ for $q > 1$. If $w^{q'} \in A_1$, Then there exists a constant $C > 0$ independent of f such that*

$$\|\mu_\Omega(f)\|_{WL_w^p} \leq C\|f\|_{H_w^p}.$$

In particular, if we take w to be a constant function, then we can get

Corollary 1.2. *Let $0 < \alpha \leq 1$, $\beta = \min\{\alpha, 1/2\}$ and $p = n/(n + \beta)$. Suppose that $\Omega \in \text{Din}_\alpha^q(S^{n-1})$ for $q > 1$. Then there exists a constant $C > 0$ independent of f such that*

$$\|\mu_\Omega(f)\|_{WL^p} \leq C\|f\|_{H^p}.$$

2 Notations and preliminaries

The definition of A_p class was first used by Muckenhoupt [14], Hunt, Muckenhoupt and Wheeden [8], and Coifman and Fefferman [2] in the study of weighted L^p boundedness of Hardy-Littlewood maximal functions and singular integrals. Let w be a nonnegative, locally integrable function defined on \mathbb{R}^n ; all cubes are assumed to have their sides parallel to the coordinate axes. We say that $w \in A_p$, $1 < p < \infty$, if

$$\left(\frac{1}{|Q|} \int_Q w(x) dx\right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx\right)^{p-1} \leq C \quad \text{for every cube } Q \subseteq \mathbb{R}^n,$$

where C is a positive constant which is independent of the choice of Q .

For the case $p = 1$, $w \in A_1$, if

$$\frac{1}{|Q|} \int_Q w(x) dx \leq C \cdot \text{ess inf}_{x \in Q} w(x) \quad \text{for every cube } Q \subseteq \mathbb{R}^n.$$

A weight function w is said to belong to the reverse Hölder class RH_r if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality holds

$$\left(\frac{1}{|Q|} \int_Q w(x)^r dx\right)^{1/r} \leq C \left(\frac{1}{|Q|} \int_Q w(x) dx\right) \quad \text{for every cube } Q \subseteq \mathbb{R}^n.$$

It is well known that if $w \in A_p$ with $1 < p < \infty$, then $w \in A_r$ for all $r > p$, and $w \in A_q$ for some $1 < q < p$. We thus write $q_w \equiv \inf\{q > 1 : w \in A_q\}$ to denote the critical index of w . Given a cube Q and $\lambda > 0$, λQ denotes the cube with the same center as Q whose side length is λ times that of Q . $Q = Q(x_0, l)$ denotes the cube centered at x_0 with side length l . For a weight function w and a measurable set E , we denote the Lebesgue measure of E by $|E|$ and set the weighted measure $w(E) = \int_E w(x) dx$.

We give the following results that will be used in the sequel.

Lemma 2.1 ([7]). *Let $w \in A_1$. Then, for any cube Q and any $\lambda > 1$, there exists an absolute constant $C > 0$ such that*

$$w(\lambda Q) \leq C \cdot \lambda^n w(Q),$$

where C does not depend on Q nor on λ .

Lemma 2.2 ([9]). *Let $r > 1$ and $A_1^r = \{w : w^r \in A_1\}$. Then we have*

$$A_1^r = A_1 \cap RH_r.$$

Lemma 2.3 ([7]). *Let $w \in A_1$. Then there exists a constant $C > 0$ such that*

$$C \cdot \frac{|E|}{|Q|} \leq \frac{w(E)}{w(Q)}$$

for any measurable subset E of a cube Q .

Given a weight function w on \mathbb{R}^n , for $0 < p < \infty$, we denote by $L_w^p(\mathbb{R}^n)$ the space of all functions satisfying

$$\|f\|_{L_w^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

We also denote by $WL_w^p(\mathbb{R}^n)$ the weighted weak L^p space which is formed by all functions satisfying

$$\|f\|_{WL_w^p} = \sup_{\lambda > 0} \lambda \cdot w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})^{1/p} < \infty.$$

We write $\mathcal{S}(\mathbb{R}^n)$ to denote the Schwartz space of all rapidly decreasing infinitely differentiable functions and $\mathcal{S}'(\mathbb{R}^n)$ to denote the space of all tempered distributions, i.e., the topological dual of $\mathcal{S}(\mathbb{R}^n)$. For any $0 < p \leq 1$, the weighted Hardy spaces $H_w^p(\mathbb{R}^n)$ can be defined in terms of maximal functions. Let φ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \varphi(x) dx = 1$. Set

$$\varphi_t(x) = t^{-n} \varphi(x/t), \quad t > 0, x \in \mathbb{R}^n.$$

We will define the maximal function $M_\varphi f(x)$ by

$$M_\varphi f(x) = \sup_{t > 0} |(\varphi_t * f)(x)|.$$

Then $H_w^p(\mathbb{R}^n)$ consists of those tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $M_\varphi f \in L_w^p(\mathbb{R}^n)$ with $\|f\|_{H_w^p} = \|M_\varphi f\|_{L_w^p}$. The real-variable theory of weighted Hardy spaces has been investigated by many authors. For example, Garcia-Cuerva [6] studied the atomic decomposition and the dual spaces of H_w^p for $0 < p \leq 1$. The molecular characterization of H_w^p for $0 < p \leq 1$ was given by Lee and Lin [11]. We refer the readers to [6, 11, 17] and the references therein for further details.

In this article, we will use Garcia-Cuerva's atomic decomposition theory for weighted Hardy spaces in [6, 17]. We characterize weighted Hardy spaces in terms of atoms in the following way.

Let $0 < p \leq 1 \leq q \leq \infty$ and $p \neq q$ such that $w \in A_q$ with critical index q_w . Set $[\cdot]$ the greatest integer function. For $s \in \mathbb{Z}_+$ satisfying $s \geq N = [n(q_w/p - 1)]$, a real-valued function $a(x)$ is called a (p, q, s) -atom centered at x_0 with respect to w (or a w -(p, q, s)-atom centered at x_0) if the following conditions are satisfied:

- (a) $a \in L_w^q(\mathbb{R}^n)$ and is supported in a cube Q centered at x_0 ;
- (b) $\|a\|_{L_w^q} \leq w(Q)^{1/q-1/p}$;
- (c) $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$ for every multi-index α with $|\alpha| \leq s$.

Theorem 2.4. *Let $0 < p \leq 1 \leq q \leq \infty$ and $p \neq q$ such that $w \in A_q$ with critical index q_w . For each $f \in H_w^p(\mathbb{R}^n)$, there exist a sequence $\{a_j\}$ of w -(p, q, N)-atoms and a sequence $\{\lambda_j\}$ of real numbers with $\sum_j |\lambda_j|^p \leq C \|f\|_{H_w^p}^p$ such that $f = \sum_j \lambda_j a_j$ both in the sense of distributions and in the H_w^p norm.*

In particular, for w equals to a constant function, we shall denote $WL_w^p(\mathbb{R}^n)$ and $H_w^p(\mathbb{R}^n)$ simply by $WL^p(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$.

Throughout this article C denotes a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence.

3 Proof of Theorem 1.1

In order to prove our main result, we shall need the following superposition principle on the weighted weak type estimates.

Lemma 3.1. *Let $w \in A_1$ and $0 < p < 1$. If a sequence of measurable functions $\{f_j\}$ satisfy*

$$w(\{x \in \mathbb{R}^n : |f_j(x)| > \alpha\}) \leq \alpha^{-p} \quad \text{for all } j \in \mathbb{Z}$$

and

$$\sum_{j \in \mathbb{Z}} |\lambda_j|^p \leq 1,$$

then we obtain that $\sum_j \lambda_j f_j(x)$ is absolutely convergent almost everywhere and

$$w\left(\left\{x \in \mathbb{R}^n : \left|\sum_j \lambda_j f_j(x)\right| > \alpha\right\}\right) \leq \frac{2-p}{1-p} \cdot \alpha^{-p}.$$

Proof. The proof of this lemma is similar to the corresponding result for the unweighted case which can be found in [16]. See also [13, p. 123]. \square

We also need the following lemma, which gives a key estimate about the kernel Ω .

Lemma 3.2. *Let $q \geq 1$. Suppose that Ω satisfies the L^q -Dini condition in Section 1. If there exists a constant $0 < \gamma \leq \frac{1}{2}$ such that $|y| < \gamma R$, then we have*

$$\left(\int_{R \leq |x| < 2R} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x)}{|x|^{n-1}} \right|^q dx \right)^{1/q} \leq C \cdot R^{n/q-n+1} \left(\frac{|y|}{R} + \int_{|y|/2R}^{|y|/R} \frac{\omega_q(\delta)}{\delta} d\delta \right),$$

where the constant $C > 0$ is independent of R and y .

Proof. By adopting the same method as in the proof of Lemma 5 in [10], we can get the desired estimate, here we omit the details. See [4] and [12]. \square

We are ready to give the proof of Theorem 1.1.

Proof. Since $w^{q'} \in A_1$, then we have $w \in A_1$ by Lemma 2.2. We now observe that for $w \in A_1$ and $p = n/(n+\beta)$, then $[n(q_w/p-1)] = [\beta] = 0$. Hence, by Theorem 2.4 and Lemma 3.1, it is enough for us to show that for any w -($p, q, 0$)-atom $a(x)$, there exists a constant $C > 0$ independent of a such that $\|\mu_\Omega(a)\|_{WL_w^p} \leq C$.

Let $a(x)$ be a w -($p, q, 0$)-atom centered at x_0 with $\text{supp } a \subseteq Q = Q(x_0, l)$, and let $Q^* = 2\sqrt{n}Q$. For any fixed $\lambda > 0$, we write

$$\begin{aligned} & \lambda^p \cdot w(\{x \in \mathbb{R}^n : |\mu_\Omega(a)(x)| > \lambda\}) \\ & \leq \lambda^p \cdot w(\{x \in Q^* : |\mu_\Omega(a)(x)| > \lambda\}) + \lambda^p \cdot w(\{x \in (Q^*)^c : |\mu_\Omega(a)(x)| > \lambda\}) \\ & = I_1 + I_2. \end{aligned}$$

By the hypothesis $w^{q'} \in A_1$, then we have $w^{q'} \in A_q$ for $1 < q < \infty$. According to Theorem B, we see that μ_Ω is bounded on $L_w^q(\mathbb{R}^n)$. Applying Chebyshev's inequality, Hölder's inequality, Lemma 2.1 and the size condition of atom a , we thus obtain

$$\begin{aligned} I_1 & \leq \int_{Q^*} |\mu_\Omega(a)(x)|^p w(x) dx \\ & \leq \left(\int_{Q^*} |\mu_\Omega(a)(x)|^q w(x) dx \right)^{p/q} \left(\int_{Q^*} w(x) dx \right)^{1-p/q} \\ & \leq \|\mu_\Omega(a)\|_{L_w^q}^p w(Q^*)^{1-p/q} \\ & \leq C \cdot \|a\|_{L_w^q}^p w(Q)^{1-p/q} \\ & \leq C. \end{aligned} \tag{3.1}$$

We turn our attention to the estimate of I_2 . First we note that if $\{x \in (Q^*)^c : |\mu_\Omega(a)(x)| > \lambda\} = \emptyset$, then the inequality

$$I_2 \leq C$$

holds trivially. Now assume that $\{x \in (Q^*)^c : |\mu_\Omega(a)(x)| > \lambda\} \neq \emptyset$. Set $Q_1^* = Q^*$ and $Q_{k+1}^* = (Q_k^*)^*, k = 1, 2, \dots$. Integrating over $Q_{k+1}^* \setminus Q_k^* (k \in \mathbb{Z}_+)$ on both sides of the inequality $\lambda < |\mu_\Omega(a)(x)|$, then we get

$$\begin{aligned}
& \lambda \cdot |Q_{k+1}^* \setminus Q_k^*| \\
& \leq \int_{Q_{k+1}^* \setminus Q_k^*} |\mu_\Omega(a)(x)| dx \\
& \leq \int_{Q_{k+1}^* \setminus Q_k^*} \left(\int_0^{|x-x_0|+\frac{\sqrt{n}}{2}l} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} a(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} dx \\
& \quad + \int_{Q_{k+1}^* \setminus Q_k^*} \left(\int_{|x-x_0|+\frac{\sqrt{n}}{2}l}^\infty \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} a(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} dx \\
& = J_1 + J_2.
\end{aligned}$$

Let us deal with the term J_1 . An application of Minkowski's integral inequality gives us that

$$\begin{aligned}
J_1 & \leq \int_{Q_{k+1}^* \setminus Q_k^*} \int_Q \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} \right| |a(y)| \left(\int_{|x-y|}^{|x-x_0|+\frac{\sqrt{n}}{2}l} \frac{dt}{t^3} \right)^{1/2} dy dx \\
& \leq C \int_{Q_{k+1}^* \setminus Q_k^*} \int_Q \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} \right| |a(y)| \cdot \left[\frac{1}{|x-y|^2} - \frac{1}{(|x-x_0|+\frac{\sqrt{n}}{2}l)^2} \right]^{1/2} dy dx.
\end{aligned}$$

For any $y \in Q$ and $x \in Q_{k+1}^* \setminus Q_k^*$, then a direct calculation shows that

$$|x-y| \sim |x-x_0| \sim |x-x_0| + \frac{\sqrt{n}}{2}l.$$

Hence

$$\frac{1}{|x-y|^2} - \frac{1}{(|x-x_0|+\frac{\sqrt{n}}{2}l)^2} \leq C \cdot \frac{l}{|x-y|^3},$$

which yields

$$J_1 \leq C \cdot \sqrt{l} \int_Q \left\{ \int_{Q_{k+1}^* \setminus Q_k^*} \left| \frac{\Omega(x-y)}{|x-y|^{n+1/2}} \right| dx \right\} |a(y)| dy.$$

For any $y \in Q$, by using Hölder's inequality and polar coordinates for integrals, we have

$$\begin{aligned}
& \int_{Q_{k+1}^* \setminus Q_k^*} \left| \frac{\Omega(x-y)}{|x-y|^{n+1/2}} \right| dx \\
& \leq \left(\int_{Q_{k+1}^* \setminus Q_k^*} \left| \frac{\Omega(x-y)}{|x-y|^{n+1/2}} \right|^q dx \right)^{1/q} \left(\int_{Q_{k+1}^* \setminus Q_k^*} 1 dx \right)^{1/q'} \\
& \leq \left(\int_{2^{k-1}l \leq |z| < 2^{k+2}l} \frac{|\Omega(z')|^q}{|z|^{nq+q/2}} dz \right)^{1/q} \cdot |Q_{k+1}^* \setminus Q_k^*|^{1/q'} \\
& \leq C \cdot \|\Omega\|_{L^q(S^{n-1})} (2^k l)^{n/q-n-1/2} |Q_{k+1}^* \setminus Q_k^*|^{1/q'}. \tag{3.2}
\end{aligned}$$

On the other hand, it follows directly from Hölder's inequality, the A_q condition and the size condition of atom a that

$$\begin{aligned} \int_Q |a(y)| dy &\leq \left(\int_Q |a(y)|^q w(y) dy \right)^{1/q} \left(\int_Q w(y)^{-q'/q} dy \right)^{1/q'} \\ &\leq C \cdot \|a\|_{L_w^q} \frac{|Q|}{w(Q)^{1/q}} \\ &\leq C \cdot \frac{|Q|}{w(Q)^{1/p}}. \end{aligned}$$

In addition, since $Q \subseteq Q_k^*$ for every $k = 1, 2, \dots$, then by Lemma 2.3, we get

$$\frac{w(Q)}{w(Q_k^*)} \geq C \cdot \frac{|Q|}{|Q_k^*|},$$

which implies

$$\int_Q |a(y)| dy \leq C \cdot \left(\frac{|Q_k^*|}{|Q|} \right)^{1/p-1} \frac{|Q_k^*|}{w(Q_k^*)^{1/p}}. \quad (3.3)$$

Summarizing the above two estimates (3.2) and (3.3) and using the fact that $p = n/(n + \beta)$, we thus obtain

$$\begin{aligned} J_1 &\leq C \cdot \sqrt{l}(2^k l)^{n/q-n-1/2} |Q_{k+1}^* \setminus Q_k^*|^{1/q'} \left(\frac{|Q_k^*|}{|Q|} \right)^{1/p-1} \frac{|Q_k^*|}{w(Q_k^*)^{1/p}} \\ &\leq C \cdot |Q_k^*|^{1/q} |Q_{k+1}^* \setminus Q_k^*|^{1/q'} \frac{1}{w(Q_k^*)^{1/p}} \cdot 2^{k(\beta-1/2)}. \end{aligned}$$

We turn to estimate the last term J_2 . In this case, it is obvious that $Q \subseteq \{y \in \mathbb{R}^n : |x - y| \leq t\}$. By the cancellation condition of atom a and Minkowski's integral inequality, we have

$$\begin{aligned} J_2 &= \int_{Q_{k+1}^* \setminus Q_k^*} \left(\int_{|x-x_0|+\frac{\sqrt{n}}{2}l}^\infty \left| \int_{|x-y|\leq t} \left[\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right] a(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} dx \\ &\leq \int_{Q_{k+1}^* \setminus Q_k^*} \int_Q \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right| |a(y)| \left(\int_{|x-x_0|+\frac{\sqrt{n}}{2}l}^\infty \frac{dt}{t^3} \right)^{1/2} dy dx \\ &\leq C \int_{Q_{k+1}^* \setminus Q_k^*} \int_Q \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right| \frac{|a(y)|}{|x-x_0|} dy dx \\ &\leq C \cdot (2^k l)^{-1} \int_Q \left\{ \int_{Q_{k+1}^* \setminus Q_k^*} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right| dx \right\} |a(y)| dy. \end{aligned}$$

Using Hölder's inequality and Lemma 3.2, we can see that for any $y \in Q$, the inner integral of the above expression is bounded by

$$\left(\int_{Q_{k+1}^* \setminus Q_k^*} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}} \right|^q dx \right)^{1/q} \left(\int_{Q_{k+1}^* \setminus Q_k^*} 1 dx \right)^{1/q'}$$

$$\begin{aligned}
&\leq C \cdot |Q_{k+1}^* \setminus Q_k^*|^{1/q'} (2^k l)^{n/q-n+1} \left(\frac{|y-x_0|}{2^k l} + \int_{|y-x_0|/2^{k+1}l}^{|y-x_0|/2^k l} \frac{\omega_q(\delta)}{\delta} d\delta \right) \\
&\leq C \cdot |Q_{k+1}^* \setminus Q_k^*|^{1/q'} (2^k l)^{n/q-n+1} \left(\frac{1}{2^k} + \frac{1}{2^{k\alpha}} \int_0^1 \frac{\omega_q(\delta)}{\delta^{1+\alpha}} d\delta \right). \tag{3.4}
\end{aligned}$$

Moreover, it follows immediately from the inequalities (3.3) and (3.4) that

$$\begin{aligned}
J_2 &\leq C \cdot (2^k l)^{n/q-n} |Q_{k+1}^* \setminus Q_k^*|^{1/q'} \left(\frac{|Q_k^*|}{|Q|} \right)^{1/p-1} \frac{|Q_k^*|}{w(Q_k^*)^{1/p}} \\
&\quad \times \left(\frac{1}{2^k} + \frac{1}{2^{k\alpha}} \int_0^1 \frac{\omega_q(\delta)}{\delta^{1+\alpha}} d\delta \right) \\
&\leq C \cdot |Q_k^*|^{1/q} |Q_{k+1}^* \setminus Q_k^*|^{1/q'} \frac{1}{w(Q_k^*)^{1/p}} \cdot 2^{k\beta} \left(\frac{1}{2^k} + \frac{1}{2^{k\alpha}} \int_0^1 \frac{\omega_q(\delta)}{\delta^{1+\alpha}} d\delta \right).
\end{aligned}$$

Clearly, for any $k = 1, 2, \dots$, we have $|Q_{k+1}^* \setminus Q_k^*| \geq |Q_k^*|$. Hence, by combining the estimates for J_1 and J_2 , we obtain

$$\begin{aligned}
\lambda &< \frac{1}{|Q_{k+1}^* \setminus Q_k^*|} \cdot \int_{Q_{k+1}^* \setminus Q_k^*} |\mu_\Omega(a)(x)| dx \\
&\leq C \cdot \frac{|Q_k^*|^{1/q}}{|Q_{k+1}^* \setminus Q_k^*|^{1/q}} \frac{1}{w(Q_k^*)^{1/p}} \cdot 2^{k(\beta-1/2)} \\
&\quad + C \cdot \frac{|Q_k^*|^{1/q}}{|Q_{k+1}^* \setminus Q_k^*|^{1/q}} \frac{1}{w(Q_k^*)^{1/p}} \cdot 2^{k\beta} \left(\frac{1}{2^k} + \frac{1}{2^{k\alpha}} \int_0^1 \frac{\omega_q(\delta)}{\delta^{1+\alpha}} d\delta \right) \\
&\leq C \cdot \left(2 + \int_0^1 \frac{\omega_q(\delta)}{\delta^{1+\alpha}} d\delta \right) \frac{1}{w(Q_k^*)^{1/p}}, \tag{3.5}
\end{aligned}$$

where the last inequality holds since $\beta \leq \min\{\alpha, 1/2\}$. Furthermore, for $p = n/(n+\beta)$, it is easy to check that

$$\lim_{k \rightarrow \infty} \frac{1}{w(Q_k^*)^{1/p}} = 0.$$

Thus, for any given $\lambda > 0$, by (3.5), we can find a maximal positive integer \mathcal{K} such that

$$\lambda < C \cdot \frac{1}{w(Q_{\mathcal{K}}^*)^{1/p}}.$$

Therefore

$$\begin{aligned}
I_2 &\leq \lambda^p \cdot \sum_{k=1}^{\mathcal{K}} w(\{x \in Q_{k+1}^* \setminus Q_k^* : |\mu_\Omega(a)(x)| > \lambda\}) \\
&\leq C \cdot \frac{1}{w(Q_{\mathcal{K}}^*)} \sum_{k=1}^{\mathcal{K}} w(Q_{k+1}^*) \\
&\leq C. \tag{3.6}
\end{aligned}$$

Combining the above inequality (3.6) with (3.1) and taking the supremum over all $\lambda > 0$, we conclude the proof of Theorem 1.1. \square

References

- [1] A. Benedek, A. P. Calderón and R. Panzone, Convolution operators on Banach space valued functions, Proc. Nat. Acad. Sci. USA, **48**(1962), 356–365.
- [2] R. R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, Studia Math, **51**(1974), 241–250.
- [3] Y. Ding, D. Fan and Y. Pan, Weighted boundedness for a class of rough Marcinkiewicz integrals, Indiana Univ. Math. J, **48**(1999), 1037–1055.
- [4] Y. Ding, M. Y. Lee and C. C. Lin, Marcinkiewicz integral on weighted Hardy spaces, Arch. Math, **80**(2003), 620–629.
- [5] Y. Ding, S. Lu and Q. Xue, Marcinkiewicz integral on Hardy spaces, Integr. Equ. Oper. Theory, **42**(2002), 174–182.
- [6] J. Garcia-Cuerva, Weighted H^p spaces, Dissertations Math, **162**(1979), 1–63.
- [7] J. Garcia-Cuerva and J. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland, Amsterdam, 1985.
- [8] R. A. Hunt, B. Muckenhoupt and R. L. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform, Trans. Amer. Math. Soc, **176**(1973), 227–251.
- [9] R. Johnson and C. J. Neugebauer, Change of variable results for A_p and reverse Hölder RH_r classes, Trans. Amer. Math. Soc, **328**(1991), 639–666.
- [10] D. S. Kurtz and R. L. Wheeden, Results on weighted norm inequalities for multipliers, Trans. Amer. Math. Soc, **255**(1979), 343–362.
- [11] M. Y. Lee and C. C. Lin, The molecular characterization of weighted Hardy spaces, J. Func. Anal, **188**(2002), 442–460.
- [12] C. C. Lin and Y. C. Lin, $H_w^p-L_w^p$ boundedness of Marcinkiewicz integral, Integr. Equ. Oper. Theory, **58**(2007), 87–98.
- [13] S. Lu, Four Lectures on Real H^p Spaces, World Scientific Publishing, River Edge, N.J., 1995.
- [14] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc, **165**(1972), 207–226.
- [15] E. M. Stein, On the functions of Littlewood-Paley, Lusin and Marcinkiewicz, Trans. Amer. Math. Soc, **88**(1958), 430–466.
- [16] E. M. Stein, M. H. Taibleson and G. Weiss, Weak type estimates for maximal operators on certain H^p classes, Rend. Circ. Mat. Palermo, Suppl. 1, **2**(1981), 81–97.

- [17] J. O. Stömberg and A. Torchinsky, Weighted Hardy spaces, Lecture Notes in Math, Vol 1381, Springer-Verlag, 1989.
- [18] A. Torchinsky and S. Wang, A note on the Marcinkiewicz integral, Colloq. Math, **60/61**(1990), 235–243.